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Fuzzy Sets in Macroscopic Quantum Systems

A. Rieckers[1]

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We express quantum properties by quantum fuzzy set functions, and these by generalized transition probabilities. The property of "being excited" is fuzzy for one two-level atoms, but shown to be crisp for infinitely many ones.

Fuzzy set theory Zadeh (1965) introduces a new type of vague properties. The interpretational questions have mostly been discussed in relationship to classical probability theory (Kosko, 1992; McNeill and Freiberger, 1994, and references therein). Our intention here is to apply some notions of fuzzy set theory to macroscopic quantum systems, where classical and quantum mechanical vagueness is combined. These systems offer interesting aspects for foundational investigations as well as for the physical realization of control systems, where fuzzy set theory has been especially useful (McNeill and Freiberger, 1994). For the following discussion we use the frame of a C*-algebraic statistical theory, which is typified by a unital C*-algebra $\mathscr{A}$ (Bratteli and Robinson, 1979). The w^*-compact, convex set of its states given by all positive, normalized linear functionals is denoted by $\mathscr{S}(\mathscr{A}) \equiv \mathscr{S}$. The statistical contents are constituted by the expectation values $\langle\rho; A\rangle$, $\rho \in \mathscr{S}(\mathscr{A})$, $A \in \mathscr{A}$, where for formal convenience we drop the restriction to the self-adjoint elements $A \in \mathscr{A}_{\text{sa}}$.

Since the set of projections in $\mathscr{A}$ may be too small, we consider the universal enveloping von Neumann algebra $\mathcal{M}(\mathscr{A}) \cong \mathscr{A}^{**}$ (which is as a Banach space isomorphic to the bidual of $\mathscr{A}$) and its projection lattice $\mathscr{P}(\mathcal{M}(\mathscr{A})) \equiv \mathscr{P}(\mathscr{A})$. It is well known (Takesaki, 1979) that $\mathscr{P}(\mathscr{A})$ is a complete orthomodular lattice, that is, a well-behaved quantum logic in the sense of J. von Neumann. In particular, the (quantum) negation $P^{\perp} = \mathbb{1} - P$, $P \in \mathscr{P}(\mathscr{A})$, has a concise, algebraic formulation in the projector language.

[1] Institut für Theoretische Physik, Universität Tübingen, D-72076 Tübingen, Germany.

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Nevertheless, it has interpretational advantages to employ a version of this proposition lattice which is more directly related to the original C*-algebraic notions, here to its state space $\mathcal{S}$. Let us, therefore, recall that the mapping

$$\mathcal{P}(\mathcal{A}) \ni P \mapsto E_P := \{\rho \in \mathcal{S}; \langle\rho; P\rangle = 1\} \subseteq \mathcal{S} \tag{1}$$

is a lattice isomorphism of $\mathcal{P}(\mathcal{A})$ onto $\mathcal{E}(\mathcal{S})$, the set of all norm-closed faces in $\mathcal{S}$, where in the latter lattice the partial ordering is given by the inclusion. Via $E_P^{\perp} = E_{P^{\perp}}$ the negation is carried over to $\mathcal{E}(\mathcal{S})$, but has for noncommutative $\mathcal{A}$ no direct formulation in the state language. The advantage to using $\mathcal{E}(\mathcal{S})$ lies, however, in considering a property E as the set of those systemic states in which E is completely actualized. This conforms with the traditional definition of a "property" as the set of objects which have it. In (1) the content of the function

$$\chi_P: \quad \mathcal{S} \ni \mathcal{S} \mapsto \chi_P(\rho) = \langle\rho; P\rangle \in [0, 1] \tag{2}$$

is only partially used. The relationship between $\mathcal{E}(\mathcal{S})$ and the set of functions

$$\mathcal{F}(\mathcal{S}) := \{\chi_E \in [0, 1]^{\mathcal{S}}; E \in \mathcal{E}(\mathcal{S}), \chi_E(\rho) = \langle\rho; P_E\rangle\} \tag{3}$$

has been investigated in Maczynsky (1973, 1974). The functions χ, χ' in $[0,1]^s$ have a natural partial ordering

$$\chi \leq \chi' \qquad \text{if} \quad \chi(\rho) \leq \chi'(\rho), \qquad \forall \rho \in \mathcal{S} \tag{4}$$

and complement

$$\chi^{\perp}(\rho) := 1 - \chi(\rho), \qquad \forall \rho \in \mathcal{S} \tag{5}$$

Since $\mathcal{F}(\mathcal{S})$ equipped with $\leq$ and $\perp$ satisfies the conditions of the "orthogonality postulate," is "complete," and "quite full," we may announce the following application of Maczynsky, (1974):

Proposition 1. $\mathcal{F}(\mathcal{S})$ is a complete orthomodular lattice, which by the mapping

$$\mathcal{F}(\mathcal{S}) \ni \chi \mapsto E_\chi = \{\rho \in \mathcal{S}; \chi(\rho) = 1\} \tag{6}$$

is ortholattice isomorphic to $\mathcal{E}(\mathcal{S})$.

According to Pykacz (1992), the version $\mathcal{F}(\mathcal{S})$ of our (quantum) logics should be considered from the point of view of fuzzy set theory, where $\mathcal{S}$ plays the role of the universe of discourse. This makes indeed the fuzziness of quantum mechanics to appear in a new light. Since (4) and (5) are the original Zadeh connections (Zadeh, 1965), they render the set of all fuzzy membership functions $[0, 1]^{\mathcal{S}}$ to a complete distributive lattice in which the

De Morgan laws are valid. The law of the excluded middle is, however, not true, and thus $[0, 1]^{\mathcal{S}}$ is not a Boolean algebra. The restriction to $\mathcal{F}(\mathcal{S})$ may change the type of fuzzy logic completely. If $\mathcal{A}$ is commutative, then $\mathcal{F}(\mathcal{S})$ is Boolean. If $\mathcal{A}$ is non-abelian, then $\mathcal{F}(\mathcal{S})$ exhibits quantum features, which in general are mixed with classical ones. The point is that in sublattices of $[0,1]^{\mathcal{S}}$ the lattice operations "meet" and "join" are no longer directly available via the membership functions in the manner of Zadeh (1965). An interesting observation by Pykacz (1992) is that for orthogonal propositions the Giles connectivities (Giles, 1976) for membership functions are applicable in the frame of Maczynsky (1974) (which covers the C*-algebraic theory).

Now we observe that our discussion in connection with Proposition 1 supports not only Kosko's statement that classical probability theory is a specialization of fuzzy set theory, but leads to the same conclusions for quantum probability. In fact, any generalized statistical theory [over a projective convex set $\mathcal{S}$ as state space (Alfsen and Shultz, 1976)] produces via the expectation functions a sublattice of $[0, 1]^{\mathcal{S}}$.

We want to supplement the extensive discussion on the relationship between fuzzy sets and statistics by demonstrating that the membership functions may be expressed in terms of "transition probabilities" which in general are not probabilities in the usual sense. The general notion of a transition probability has been introduced by Cantoni (1975) in the frame of Mackey systems and has been axiomatically founded in a clear and appealing manner in Gudder (1978). For the C*-algebraic theory its equivalence to Uhlmann's transition probability (Uhlmann, 1976), which we use in the sequel, has been shown in Araki and Raggio (1982).

Definition 2 (Uhlmann, 1976). For two states $\varphi,\psi \in \mathcal{S}$ their (*a priori*) transition probability is defined as

$$T_{\mathcal{A}}(\varphi, \psi) := \sup_{\Pi,\Phi^{\Pi},\Psi^{\Pi}} |(\Phi^{\Pi}|\Psi^{\Pi})|^2 \tag{7}$$

where the supremum runs over all representations Π of $\mathcal{A}$, in which φ,ψ both have vector representatives Φ^{Π} and Ψ^{Π}, which are also varied in (7). Here $(\cdot|\cdot)$ denotes the scalar product in the representation Hilbert space of Π.

In (7) the analogy to the transition probability of traditional Hilbert space quantum mechanics for two pure states is suggestive. Observe, however, that here the states φ,ψ may be mixed, even classically mixed (nonfactorial), and that for commutative $\mathcal{A}$ we arrive at the classical transition measure of Kakutani (1948).

For calculations it is advantageous to avoid the supremum in (7). In Gerisch *et al.* (1996) it is shown that for given Π (in which φ,ψ both have vector representatives) and given Φ^{Π} there exists a distinguished Ψ^{Π} with

$$T_{\mathscr{A}}(\varphi, \psi) = |(\Phi^{\Pi}|\Psi^{\Pi})|^2 \tag{8}$$

Now we come to the decisive result.

Proposition 3. For any membership functions $\chi_E \in \mathscr{F}(\mathscr{S})$ it holds that

$$\chi_E(\rho) = \sup_{\psi \in E} T_{\mathscr{A}}(\rho, \psi) \tag{9}$$

Proof. The proof uses Hilbert space geometry in the universal representation space of $\mathscr{A}$ (Rieckers and Zanzinger, 1996). ■

If in (9) ψ is pure and E equal to the singleton face $\{\psi\}$, then the membership function $\chi_E(\rho)$ coincides with the transition probability. Also, for the set of factorial states $\mathscr{S}_f$ the relation (9) has a clarifying consequence for classical properties.

Proposition 4. If the property $E \in \mathscr{E}(\mathscr{S})$ is classical (split face, folium), then $\chi_E|_{\mathscr{S}f}$ is crisp (has only the values 0 and 1). If $E \subseteq E(\mathscr{S}_f)$, then the reverse implication holds as well. Here $E(\mathscr{S}_f)$ denotes the smallest folium containing all factor states.

Proof. The proof is easy with results of Gerisch *et al.* (1996). ■

For quasi-local C*-algebras there exists a net $\Lambda \mapsto \mathscr{A}_\Lambda$, $\Lambda \in I$, of local algebras such that $\mathscr{A} = \overline{\cup_I \mathscr{A}_\Lambda}^{\|\cdot\|}$. These algebras give rise to continuously varying classical properties. Let us, for example, consider a macroscopic amount of two-level atoms. Then I consists of the finite subsets Λ of $\mathbb{N}$, the numbering of the atoms. For the ith atoms we have the C*-algebraic theory

$$\mathscr{A}_i \cong \mathbb{M}_2, \qquad \mathscr{S} \cong \mathscr{T}^1_+(\mathbb{C}^2) \cong \mathscr{B}^3 \tag{10}$$

that is, the algebra of 2 × 2 matrices with its state space affine isomorphic to the ball in $\mathbb{R}^3$ with radius 1/2. For every $\underline{x} \in \mathscr{B}^3$ there is a unique density operator $\rho_{\underline{x}} \in \mathscr{T}^1_+(\mathbb{C}^2)$ with $\mathrm{tr}[\rho\underline{\sigma}] = \underline{x}$, where $\underline{\sigma}$ is the vector of the three Pauli matrices. The faces of $\mathscr{B}^3$ are the sets $\emptyset$, the singletons $\{\underline{x}\}$, where $\|\underline{x}\| = 1/2$, and $\mathscr{B}^3$. The corresponding membership functions $\chi \in \mathscr{F}(\mathscr{B}^3)$ are the constant zero, the unique affine functions $\chi_{\underline{x}}: \mathscr{B}^3 \mapsto \mathbb{R}$, with $\chi_{\underline{x}}(\underline{x}) = 1$, and $\chi_{\underline{x}}(-\underline{x}) = 0$, and the constant unity, respectively. If $\underline{x}, \underline{y} \in \mathscr{B}^3$ with $\|\underline{x}\| = \|\underline{y}\| = 1/2$, and $\underline{x} \neq \underline{y}$, then $\chi_{\underline{x}} \wedge \chi_{\underline{y}} \equiv 0$, since no other element in $\mathscr{F}(\mathscr{B}^3)$ is pointwise smaller than x and than y.

According to Dicke (1954), the state properties for systems of two-level atoms, which are relevant for the radiated photon states, are the cooperation number $s \in [0, 1]$ and the excitation number $\gamma \in [0, 1]$. For one two-level atom they are defined by

$$s = 2\|\bar{x}\|, \qquad \gamma = x_3 + \frac{1}{2}, \qquad \bar{x} \in \mathcal{B}^3 \tag{11}$$

The full excitation property is given by the singleton { (0, 0, 1/2) }. But also the other states $\rho_{\bar{x}}$, $\bar{x} \in \mathcal{B}^3$, all have a partial excitation, with the only exception of (0, 0, −1/2). The membership function $\chi_{\bar{x}}$, $\bar{x} = (0, 0, 1/2)$, for the property "excitation" spreads over all of $\mathcal{B}^3$. If one prepares any state $\chi_{\bar{x}}$, $\bar{x} \neq (0, 0, -1/2)$, one has a finite likelihood to find the atom excited. Here "likelihood" is meant as a circumlocution of "transition probability" (Gudder, 1978).

To study a macroscopic (radiating) system of two-level atoms we choose the C*-algebraic theory

$$\mathcal{A} := \bigotimes_{\mathbb{N}} \mathcal{A}_i, \qquad \mathcal{S} = \mathcal{S}(\mathcal{A}) \tag{12}$$

that is, the infinite-atom limit. For every $\bar{x} \in \mathcal{B}^3$ we form the set of states

$$E_{\bar{x}} = \left\{ \varphi \in \mathcal{S};\ \lim_{\Lambda} \langle \varphi; \sum_{i \in \Lambda} \bar{\sigma} / |\Lambda| \rangle = \bar{x} \right\} \tag{13}$$

where $|\Lambda|$ is the cardinality of $\Lambda \subset \mathbb{N}$.

$E_{\bar{x}}$ is convex, norm-closed and stable against local perturbations. [Skip a finite part in the sum of (13)!] Thus $E_{\bar{x}}$ is a folium (Haag *et al.*, 1970), that is, a classical property in $\mathcal{E}(\mathcal{S})$. The restriction of $\chi_{E_{\bar{x}}}$ to $\mathcal{S}_f$ is therefore crisp. If one prepares an ensemble of lasers pumped to the excitation degree $\gamma = \gamma(\bar{x})$, each exemplar "has" this degree if it is in a factorial state. *All* subsets of $\mathcal{B}^3$ are now (classical) properties (and not only the few previous ones of one two-level atom).

The interaction with the photon field, as described in terms of a Dicke model, transfers the cooperation and excitation properties to the outgoing radiation states (Honegger and Rieckers, 1994). Their spatial distribution depends on the wave functions for the eigenstates of the two-level atom. Their overall intensity is proportional to

$$I(\bar{x}) = \left[\left(\frac{s}{2}\right)^2 - \left(\gamma - \frac{1}{2}\right)^2 \right] \tag{14}$$

The outgoing radiation is completely coherent if $I(\bar{x})$ is given a sharp value by the atomic states preparation. The control system "atom plus radiation" works mainly in terms of classical properties. This somewhat explains the usefulness of the semiclassical approximation in laser physics. In the Josephson microwave radiation the macroscopic phase dynamics is nonclassical (Hofmann and Rieckers, 1996). In this case the nature of quantum fuzziness, integrated into an optoelectronic control system, may be studied on the macroscopic scale.

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